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# Consistency of spin-1 theories in external electromagnetic fields

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**Abstract.** It is known that while the spin-1 (Proca) theory with minimal electromagnetic interaction is marred by the Corben–Schwinger anomaly, the inclusion of *specific* types of 'anomalous' interactions leads to other difficulties such as non-causality of propagation, and the energy spectrum (in homogeneous magnetic fields of sufficient strength) becoming partially non-real. In this paper we investigate whether, by making the anomalous terms (introduced into the Duffin–Kemmer–Petiau equation) sufficiently general, it is possible to escape both these inconsistencies. The answer, unfortunately, turns out to be in the negative. We comment briefly on other formulations for spin-1.

## 1. Introduction

Investigations of relativistic wave equations for spin $>\frac{1}{2}$ in recent years have revealed a variety of anomalies arising when interactions with external fields are introduced (Velo and Zwanziger 1969a, 1969b; Wightman 1971, 1973; other references may be traced from Mathews *et al* 1976). The classic work of Johnson and Sudarshan (1961) on the quantisation of spin-$\frac{3}{2}$ fields minimally coupled to external electromagnetic fields was the first to bring to light serious inconsistencies. It has since been discovered that even at the level of $c$-number fields one or other of several types of problem may arise: non-causal propagation, revealed in Velo and Zwanziger's (1969a) study of spin $\frac{3}{2}$ fields coupled to external electromagnetic fields; complex values of energy eigenvalues, found by Tsai *et al* (Tsai and Yildiz 1971, Goldman *et al* 1971, Tsai 1971) in their study of spin-1 particles with non-minimal electromagnetic interaction and investigated extensively by Mathews and collaborators (Mathew 1974, Seetharaman *et al* 1975, Prabhakaran 1975, Mathews *et al* 1976) and changes in the number of degrees of freedom of the field considered (Federbush 1961, Velo and Zwanziger 1969b). These discoveries have led to investigations on a number of wave equations for spins 1, $\frac{3}{2}$ and 2, with a variety of *specific* interactions with a view of determining to what extent they are free of (one or more of) the above difficulties (Schroer *et al* 1970, Minkowski and Seiler 1971, Velo 1972, Shamaly and Capri 1972, Jenkins 1972, Hagen 1972, Singh 1973, Capri and Shamaly 1976).

Surprisingly, however, there has been no attempt, even in the relatively simple case of spin-1 particles interacting with external electromagnetic fields, to find out whether all the above types of inconsistency can be overcome by making the interaction sufficiently general†. The need for such a study in the context of spin-1 theories is all the

† Shamaly and Capri 1972 gives an analysis of the Rarita–Schwinger equation with general non-minimal terms.

greater because the pattern of results obtained so far is somewhat confusing. For example, the equations employing antisymmetric tensor wavefunctions (Shay and Good 1969, Takahashi and Palmer 1970) lead to complex energy modes for charged spin-1 particles in homogeneous magnetic fields (HMF) if the interaction is such as to correspond to a $g$-factor different from unity, the 'normal' value (Prabhakaran *et al* 1975). The same is true of the familiar Proca equation. However, while the former are causal only for this same coupling ($g = 1$) (Prabhakaran and Seetharaman 1973), the latter remains causal even if an anomalous magnetic moment of arbitrary strength is added. Apart from the resolution of such apparent discrepancies, one needs to know whether there is *any* combination of the various possible non-minimal interactions which would make the spin-1 theory both causal and free of complex energy modes in HMF. (The possibility of interactions causing changes in the number of degrees of freedom does not arise in the spin-1 case).

For the purpose of our study, the Duffin–Kemmer–Petiau (DKP) equation (Duffin 1936, Kemmer 1939, Petiau 1938)—the wavefunction in which is made up of a four vector and an antisymmetric tensor—is most convenient. Since it is a first order equation there is no ambiguity in the introduction of the minimal interaction through the usual replacement $p_\mu \to \pi_\mu \equiv p_\mu - eA_\mu$. (This is to be contrasted with second order equations, say of Takahashi and Palmer, wherein one can replace $p_\mu p_\nu \to \pi_\mu \pi_\nu$ or $a\pi_\mu\pi_\nu + (1-a)\pi_\nu\pi_\mu$). To introduce anomalous terms, one has to construct all possible tensors out of the Duffin–Kemmer $\beta$-matrices and couple them suitably to similar tensors constructed out of the electromagnetic field. The set of all possible tensors and their parity properties have been classified in a systematic way by Glass (1971) and hence the construction of a very general anomalous term is straightforward. We shall study the question of causality of propagation of the wavefunction satisfying the DKP equation including such a general interaction term, and also solve the equation explicitly in the case that the external field is a constant HMF, to determine the nature of the energy spectrum.

The plan of the paper is as follows: In the next section we briefly review the DKP equation for spin-1 and Glass's classification of the tensors associated with the theory. With the help of a convenient representation of the $\beta_\mu$ (also given in § 2) we solve in § 3 the resulting generalised DKP equation, in the special case when the external field is a constant HMF. The conditions for the energy spectrum to be wholly real (irrespective of the strength of the magnetic field) are obtained. The problem of causality is investigated in § 4, and the conditions for propagation to be causal are derived. The structure of the interaction terms in the causal theory when reduced to the Proca and the Shay–Good forms is discussed in § 5. The final section provides a discussion of the results obtained.

## 2. DKP equation for spin-1 and the algebra of $\beta$-matrices

The matrices $\beta_\mu$ in the DKP equation

$$(\beta_\mu \partial_\mu + m)\psi = 0 \tag{1}$$

obey the relation

$$\beta_\mu\beta_\nu\beta_\rho + \beta_\rho\beta_\nu\beta_\mu = \beta_\mu\delta_{\nu\rho} + \beta_\rho\delta_{\nu\mu} \tag{2}$$

and $\psi$ is a ten component wavefunction which transforms according to the reducible representation $D(0,1)\oplus D(1,0)\oplus D(\frac{1}{2},\frac{1}{2})$ of the Lorentz group. It is well known that equation (2) leads to a finite matrix algebra having 126 linearly independent elements and three irreducible representations of dimension 10, 5 and 1, and that the 10-dimensional and 5-dimensional representations, when used in equation (1), provide characterisations of particles of spin-1 and 0 respectively. Glass (1971) has recently constructed a basis for the algebra in terms of tensors constructed from the $\beta_\mu$. In this basis, the 100 elements pertinent to the 10-dimensional (spin-1) representation appear as components of 15 irreducible tensors. They are displayed in table 1, wherein we have used the following definitions:

$$B=\beta_\mu\beta_\mu, \qquad \Gamma=\epsilon_{\kappa\lambda\mu\nu}\beta_\kappa\beta_\lambda\beta_\mu\beta_\nu; \qquad \alpha_\mu=\epsilon_{\mu\lambda\kappa\nu}\beta_\lambda\beta_\kappa\beta_\nu;$$

$$\sigma_{\mu\nu}=[\beta_\mu,\beta_\nu]; \qquad B_{\mu\nu}=\{\beta_\mu,\beta_\nu\}; \qquad S_{\mu\nu}=\{\beta_\mu,\alpha_\nu\}+\{\beta_\nu,\alpha_\mu\}.$$

$$T_{\mu\nu\rho}=\mathrm{i}\{\beta_\mu,\sigma_{\nu\rho}\}+\tfrac{2}{3}\epsilon_{\mu\nu\rho\kappa}\alpha_\kappa-\tfrac{1}{3}\mathrm{i}\delta_{\mu\nu}[B,\beta_\rho]+\tfrac{1}{3}\mathrm{i}\delta_{\mu\rho}[B,\beta_\nu];$$

$$\begin{aligned}Y_{\kappa\lambda\mu\nu}={}&\{\sigma_{\kappa\mu},\sigma_{\lambda\nu}\}-\{\sigma_{\kappa\nu},\sigma_{\lambda\mu}\}-\tfrac{1}{2}\delta_{\kappa\mu}\{\sigma_{\rho\nu},\sigma_{\rho\lambda}\}-\tfrac{1}{2}\delta_{\lambda\nu}\{\sigma_{\rho\mu},\sigma_{\rho\kappa}\}\\&+\tfrac{1}{2}\delta_{\kappa\nu}\{\sigma_{\rho\mu},\sigma_{\rho\lambda}\}+\tfrac{1}{2}\delta_{\lambda\mu}\{\sigma_{\rho\nu},\sigma_{\rho\kappa}\}+\tfrac{1}{6}(\delta_{\kappa\mu}\delta_{\lambda\nu}-\delta_{\kappa\nu}\delta_{\lambda\mu})\{\sigma_{\rho\alpha},\sigma_{\rho\alpha}\}.\end{aligned} \tag{3}$$

(In the above the convention of summation over repeated indices is assumed). We use these tensors now to generalise the DKP equation to include all possible non-minimal as well as minimal electromagnetic interaction of the spin-1 particle.

**Table 1.** Hermitian tensor basis for PDK spin 1 theory

| Representation of the Lorentz group | Independent Hermitian tensors | No. of elements |
|---|---|---|
| $D_+^{(0,0)}$ (Scalar) | $1; B$ | 2 |
| $D_-^{(0,0)}$ (Pseudo scalar) | $\Gamma$ | 9 |
| $D_+^{(1/2,1/2)}$ (Vector) | $\beta_\mu$: $\mathrm{i}[B,\beta_\mu]$ | 8 |
| $D_-^{(1/2,1/2)}$ (Pseudo vector) | $\alpha_\mu$: $\mathrm{i}[B,\alpha_\mu]$ | 8 |
| $D_+^{(1,1)}$ (Symmetric traceless tensor) | $B_{\mu\nu}-\frac{1}{2}g_{\mu\nu}B$: $\{B,B_{\mu\nu}\}-g_{\mu\nu}B^2$ | 18 |
| $D_-^{(1,1)}$ (Symmetric traceless pseudo tensor) | $S_{\mu\nu}$ | 9 |
| $D^{(1,0)}\oplus D^{(0,1)}$ (antisymmetric tensor) | $\mathrm{i}[\beta_\mu,\beta_\nu]$: $\mathrm{i}\{B,[\beta_\mu,\beta_\nu]\}$ | 12 |
| $D^{(3/2,1/2)}\oplus D^{(1/2,3/2)}$ | $T_{\mu\nu\rho}$: $\mathrm{i}\{B,T_{\mu\nu\rho}\}$ | 32 |
| $D^{(2,0)}\oplus D^{(0,2)}$ | $Y_{\kappa\lambda\mu\nu}$ | 10 |

## 3. Coupling to an external electromagnetic field

Taking into account possible anomalous couplings to the external field in addition to the minimal coupling (effected by the usual replacement $\partial_\mu\to\partial_\mu-\mathrm{i}eA_\mu$) we can write the equation of motion in the form

$$[\mathrm{i}\beta_\mu\pi_\mu+m+R(x)]\psi(x)=0 \tag{4}$$

with $\pi_\mu=-\mathrm{i}\,\partial_\mu-eA_\mu$. Here $R(x)$ represents the anomalous couplings which, for reasons of gauge invariance, can depend on the $A_\mu$ only through $F_{\mu\nu}\equiv\partial_\mu A_\nu-\partial_\nu A_\mu$. $R(x)$ can be expanded in terms of the tensors referred to in the last section, the coefficients in the expansions being functions of $F_{\mu\nu}$. It can be verified that in view of the antisymmetry of $F_{\mu\nu}$, the most general form of $R$ is

$$R = (\mathrm{i}e/2m)(g_1\sigma_{\mu\nu} + g_2\{B, \sigma_{\mu\nu}\})F_{\mu\nu} + (e^2/2m^3)[g_3(B_{\mu\nu} - \tfrac{1}{2}\delta_{\mu\nu}B) + g_4(\{B, B_{\mu\nu}\} - \delta_{\mu\nu}B^2)]F_{\mu\alpha}F_{\alpha\nu} + (e^2/2m^3)g_5 Y_{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} + (e^2/2m^3)g_6 BF_{\alpha\beta}F_{\beta\alpha}. \tag{5}$$

In the above we have assumed that the interaction is parity conserving and that there are no derivative couplings†. Factors of $m$ have been included explicitly along with the coupling parameters $g_1 \ldots$ for dimensional reasons. The parameters themselves may be constants or functions of invariants constructed from $F_{\mu\nu}$. (For convenience of reference we shall talk of the first two terms in (5)—involving $g_1$ and $g_2$—as terms linear in $F_{\mu\nu}$ and the others as quadratic terms, though such a description is clearly valid only if the $g$'s themselves are merely constants). In the rest of the paper we shall be concerned with equation (4) with $R$ given by equation (5).

For explicit calculations, we shall employ the following representation of the $\beta$-matrices (Seetharaman *et al* 1970):

$$\beta_j = \begin{bmatrix} 0 & 0 & 0 & -e_j \\ 0 & 0 & -\mathrm{i}s_j & 0 \\ 0 & \mathrm{i}s_j & 0 & 0 \\ -e_j^\dagger & 0 & 0 & 0 \end{bmatrix} \qquad \beta_4 = \begin{bmatrix} 0 & 0 & -\mathrm{i} & 0 \\ 0 & 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{6}$$

Here the $10 \times 10$ matrices are written in partitioned form, with the ten rows (and columns) grouped into $3+3+3+1$. The $s_j (j = 1, 2, 3)$ are the spin-1 matrices with elements

$$(s_j)_{kl} = -\mathrm{i}\epsilon_{jkl} \tag{7a}$$

and the $e_j$ are $3 \times 1$ matrices with

$$(e_j)_k = \delta_{jk} \tag{7b}$$

The ten component wavefunction $\psi$ will also be partitioned conformably into parts

$$\psi_1 = \begin{bmatrix} \psi_{\mathrm{I}} \\ \psi_{\mathrm{II}} \\ \psi_{\mathrm{III}} \end{bmatrix}; \qquad \psi_2 = \begin{bmatrix} \psi_{\mathrm{IV}} \\ \psi_{\mathrm{V}} \\ \psi_{\mathrm{VI}} \end{bmatrix}; \qquad \psi_3 = \begin{bmatrix} \psi_{\mathrm{VII}} \\ \psi_{\mathrm{VIII}} \\ \psi_{\mathrm{IX}} \end{bmatrix}; \qquad \psi_4 = \psi_{\mathrm{X}}. \tag{8}$$

## 4. Solutions in a homogeneous magnetic field

We shall now obtain solutions of equation (4) with $R$ given by (5), in the case when $A_\mu$ corresponds to an external magnetic field (homogeneous and static) in the $z$ direction, so that

$$F_{12} = -F_{21} = \mathscr{H} \tag{9}$$

† Parity conserving interaction terms can be constructed from the pseudo scalar $\Gamma$ and the pseudo symmetric tensor $S_{\mu\nu}$ by coupling them to expressions involving the dual tensor $\hat{F}_{\mu\nu} = (\mathrm{i}/2)\epsilon_{\mu\nu\kappa\lambda}F_{\kappa\lambda}$. However such terms necessarily contain the factor $\hat{F}_{\mu\nu}F_{\mu\rho}$ which vanishes in the two special cases of interest, viz pure electric and pure magnetic fields. Hence the addition of such terms does not alter the results of the analysis to be presented below.

and all other components of $F_{\mu\nu}$ vanish. In this special case, $R$ of equation (5) has the simple form

$$R = m\begin{bmatrix} R_1 & 0 & 0 & 0 \\ 0 & R_2 & 0 & 0 \\ 0 & 0 & R_3 & 0 \\ 0 & 0 & 0 & R_4 \end{bmatrix} \tag{10}$$

with

$$\begin{aligned} R_1 &= \alpha_1\xi^2 + \alpha_2\xi\varsigma_3 + \alpha_3\xi^2\varsigma_3^2 \\ R_2 &= \rho_1\xi^2 + \rho_2\xi\varsigma_3 + \rho_3\xi^2\varsigma_3^2 \\ R_3 &= \sigma_1\xi^2 + \sigma_2\xi\varsigma_3 + \sigma_3\xi^2\varsigma_3^2 \\ R_4 &= \sigma_1\xi^2 \end{aligned} \tag{11}$$

where

$$\xi = e\mathscr{H}/m^2.$$

The coefficients appearing in (11) are certain linear combinations† of the $g_i$:

$$\begin{aligned} &\alpha_1 = g_3 + 4g_4 + \tfrac{8}{3}g_5 - 2g_6 && \rho_1 = -(g_3 + 4g_4) + \tfrac{8}{3}g_5 - 2g_6 \\ &\alpha_2 = -(g_1 + 4g_2) && \rho_2 = -(g_1 + 4g_2) \\ &\alpha_3 = -(g_3 + 4g_4 + 4g_5) && \rho_3 = g_3 + 4g_4 - 4g_5 \\ &\sigma_1 = -(\tfrac{1}{2}g_3 + 3g_4 + 3g_6) && \sigma_2 = -(g_1 + 6g_2) \\ &\sigma_3 = (g_3 + 6g_4). \end{aligned} \tag{12}$$

The equations of motion for the parts $\psi_i$ of $\psi$ now become

$$\begin{aligned} &m(1 + \alpha_1\xi^2 + \alpha_2\xi\varsigma_3 + \alpha_3\xi^2\varsigma_3^2)\psi_1 + \pi_4\psi_3 - \mathrm{i}\boldsymbol{e}\cdot\boldsymbol{\pi}\psi_4 = 0 \\ &m(1 + \rho_1\xi^2 + \rho_2\xi\varsigma_3 + \rho_3\xi^2\varsigma_3^2)\psi_2 + \boldsymbol{s}\cdot\boldsymbol{\pi}\psi_3 = 0 \\ &-\pi_4\psi_1 - \boldsymbol{s}\cdot\boldsymbol{\pi}\psi_2 + m(1 + \sigma_1\xi^2 + \sigma_2\xi\varsigma_3 + \sigma_3\xi^2\varsigma_3^2)\psi_3 = 0 \\ &-\mathrm{i}\boldsymbol{e}^\dagger\cdot\boldsymbol{\pi}\psi_1 + m(1 + \sigma_1\xi^2)\psi_4 = 0. \end{aligned} \tag{13}$$

To solve these equations we apply the method employed in our earlier papers. (Mathews 1974, Prabhakaran *et al* 1975). It rests on the observation that with the electromagnetic field specified as above, the operators

$$a = (2e\mathscr{H})^{-1/2}\pi_+ \quad \text{and} \quad a^+ = (2e\mathscr{H})^{-1/2}\pi_- \tag{14a}$$

($\pi_\pm = \pi_1 \pm \mathrm{i}\pi_2$) obey the commutation relations of the ladder operators of the harmonic oscillator

$$[a, a^+] = 1. \tag{14b}$$

Of course, $a$ and $a^+$ commute with the spin operator $\varsigma$ and also, in the present case, with $\pi_3$. To take advantage of this fact we define a basis consisting of simultaneous

† Since then are only six $g_i$'s, there must exist three relations among the nine coefficients defined in (12). These are easily seen to be $\alpha_2 = \rho_2$, $\alpha_1 + \alpha_3 = \rho_1 + \rho_3$ and $3(\alpha_1 + \rho_1) + 2(\alpha_3 + \rho_3) = 4\sigma_1 + 2\sigma_3$.

eigenstates† $|n, \mu\rangle$ of the number operator $N \equiv a^+a$ and the third component $s_3$ of spin:

$$\begin{aligned} N|n, \mu\rangle &= n|n, \mu\rangle, \qquad & n &= 0, 1, 2, \ldots \\ a|n, \mu\rangle &= n^{1/2}|n-1, \mu\rangle, & \mu &= 1, 0, -1 \\ a^\dagger|n, \mu\rangle &= (n+1)^{1/2}|n+1, \mu\rangle \\ \sigma_3|n, \mu\rangle &= \mu|n, \mu\rangle. \end{aligned} \tag{15}$$

We expand the $\psi_i$ in terms of these basis states. Further, as we are interested in stationary states with time-dependence $e^{-iEt}$, we replace $\pi_4 \equiv p_4$ in (13) by $iE$. A close look at equation (13) then reveals that the $\psi_i$ have the following structure:

$$\begin{aligned} \psi_1 &= c_1|n, 0\rangle + c_2|n+1, 1\rangle + c_3|n-1, -1\rangle \\ \psi_2 &= id_1|n, 0\rangle + id_2|n+1, 1\rangle + id_3|n-1, -1\rangle \\ \psi_3 &= if_1|n, 0\rangle + if_2|n+1, 1\rangle + if_3|n-1, 1\rangle \\ \psi_4 &= id_4|n, 0\rangle. \end{aligned} \tag{16}$$

The coefficients $c_i$, $d_i$ and $f_i$ here are parameters which are as yet undetermined. (The notation suppresses their $n$-dependence.) For example, with $n = 0$ one has

$$\begin{aligned} \psi_1 &= c_1|0, 0\rangle + c_2|1, 1\rangle; \qquad & \psi_2 &= id_1|0, 0\rangle + id_2|1, 1\rangle \\ \psi_3 &= if_1|0, 0\rangle + if_2|1, 1\rangle; & \psi_4 &= id_4|0, 0\rangle. \end{aligned} \tag{17}$$

Substitution of the forms (16) in equation (13) followed by the use of (15) leads to a set of linear equations in the $c_i$, $d_i$ and $f_i$ when the coefficients of the linearly independent states on the LHS are equated to zero. Equations (16) give solutions also for $n = 0$ and $n = -1$ but with the understanding that one must drop those terms in which a negative number appears in the place of the eigenvalue of $N$. The calculations are straightforward though laborious. The form of the equations can be simplified somewhat by introducing the definitions

$$\begin{aligned} \delta_1 &= \frac{1}{\xi^2(1+\rho_1\xi^2)}; \qquad \delta_2 = \frac{\rho_2}{\rho_2^2\xi^2 - [1+(\rho_1+\rho_3)\xi^2]^2} \\ \delta_3 &= \frac{\rho_3[1+(\rho_1+\rho_3)\xi^2] - \rho_2^2}{(1+\rho_1\xi^2)[\rho_2^2\xi^2 - \{1+(\rho_1+\rho_3)\xi^2\}^2]}; \qquad \epsilon = E/m. \end{aligned} \tag{18}$$

One obtains thus the following equations for the coefficients appearing in (16):

$$-\epsilon c_1 + [1 + \sigma_1\xi^2 + \delta_2\xi^2 + (\delta_1+\delta_3)\xi^2(2n+1)\xi]f_1 = 0 \tag{19a}$$

$$(1+\alpha_1\xi^2)c_1 - \epsilon f_1 = 0 \tag{19b}$$

$$-\epsilon c_2 + [1 + \sigma_2\xi + (\sigma_1+\sigma_3)\xi^2 + \delta_1\xi^2(2n+1)\xi]f_2 + \delta_1\xi^2[n(n+1)]^{1/2}\xi f_3 = 0 \tag{20a}$$

$$-\epsilon c_3 + \delta_1\xi^2[n(n+1)]^{1/2}\xi f_2 + [1 - \sigma_2\xi + (\sigma_1+\sigma_3)\xi^2 + \delta_1\xi^2 n\xi]f_3 = 0 \tag{20b}$$

$$\left[1 + \alpha_2\xi + (\alpha_1+\alpha_3)\xi^2 + \frac{(n+1)\xi}{1+\sigma_1\xi^2}\right]c_2 - \frac{[n(n+1)]^{1/2}\xi}{1+\sigma_1\xi^2}c_3 - \epsilon f_2 = 0 \tag{20c}$$

† In what follows we consider only the case where the eigenvalue of $\pi_3 = 0$. Generalisation to non-zero eigenvalues of $\pi_3$ is straightforward and presents only non-essential complications.

$$-\frac{[n(n+1)]^{1/2}\xi}{1+\sigma_1\xi^2}c_2+\left[(\alpha_1+\alpha_3)\xi^2+\frac{n\xi}{1+\sigma_1\xi^2}-\alpha_2\xi+1\right]c_3-\epsilon f_3=0 \quad (20d)$$

$$d_1=-\delta_1\xi^2\{[(n+1)\xi]^{1/2}f_2+(n\xi)^{1/2}f_3\}, \quad (21a)$$

$$d_2=-[(n+1)\xi]^{1/2}[(\delta_1+\delta_3)\xi^2+\delta_2\xi]f_1, \quad (21b)$$

$$d_3=-(n\xi)^{1/2}[(\delta_1+\delta_3)\xi^2-\delta_2\xi]f_1, \quad (21c)$$

$$d_4=\frac{1}{1+\sigma_1\xi^2}\{(n\xi)^{1/2}c_3-[(n+1)\xi]^{1/2}c_2\}. \quad (21d)$$

Equations (19) show that $c_1$ and $f_1$ can have non-zero values only if

$$\epsilon^2=(1+\alpha_1\xi^2)\{1+[\sigma_1+\delta_2+(\delta_1+\delta_3)(2n+1)\xi]\xi^2\}. \quad (22)$$

Similarly, equations (20) will have non-zero solutions, provided $\epsilon$ is such as to satisfy the following quadratic equation in $\epsilon^2$:

$$\epsilon^4-p\epsilon^2+q=0 \quad (23)$$

where

$$p=\sigma_+\alpha_++\sigma_-\alpha_-+\delta_1\xi^2[(n+1)\alpha_++n\alpha_-]+\kappa\xi[(n+1)\sigma_++n\sigma_-]+\kappa\delta_1\xi^4 \quad (24a)$$

and

$$q=\{\sigma_+\sigma_-+\delta_1\xi^2[(n+1)\sigma_-+n\sigma_+]\}\{\alpha_+\alpha_-+\kappa\xi[(n+1)\alpha_-+n\alpha_+]\}. \quad (24b)$$

The following notation has been used in the above for ease of writing:

$$\begin{aligned}\alpha_\pm&=[1+(\alpha_1+\alpha_3)\xi^2]\pm\alpha_2\xi\\ \sigma_\pm&=[1+(\sigma_1+\sigma_3)\xi^2]\pm\sigma_2\xi\\ \kappa&=\frac{1}{1+\sigma_1\xi^2}.\end{aligned} \quad (25)$$

Equations (22) and (23) determine the conditions to be satisfied in order that all solutions for the parameter $E$ be real, i.e. $\epsilon^2=E^2/m^2>0$. Since reality is required for all values of $n$ and $\xi$, one can readily obtain a number of necessary conditions by considering the limits of very large $\xi$, very large or small $n$ etc, and make use of them in analysing further the case of general $n$ and $\xi$. One finds in this manner that the necessary and sufficient conditions for reality of $E$ for all $n$ and $\xi$ are:

$$\alpha_1, \alpha_3, \rho_1, \rho_3, \sigma_1, \sigma_3\geqslant 0; \quad (26a)$$

and

$$\begin{aligned}&(\alpha_1+\alpha_3)\geqslant\tfrac{1}{4}\alpha_2^2; \qquad (\rho_1+\rho_3)\geqslant\tfrac{1}{4}\rho_2^2\\ &(\sigma_1+\sigma_3)\geqslant\tfrac{1}{4}\sigma_2^2.\end{aligned} \quad (26b)$$

To see what these conditions imply, we refer back to the definitions (12) of the quantities appearing in (26). On inspection one sees that equations (26$a$) can be readily satisfied. Equations (26$b$) however have an important consequence. They cannot be satisfied if the coefficients $g_1$, $g_2$ of the linear terms in (5) alone are non-zero. Hence, if the interaction involves linear terms in $F_{\mu\nu}$ it must also contain quadratic terms if the energy spectrum is to be completely real. We shall see at the end of the next section that this result makes it impossible to have any non-minimal interaction consistent with causality and reality of the energy values $E$.

## 5. Propagation of DKP waves in an external electromagnetic field

To investigate the question of causality of propagation, we shall adopt the familiar method of characteristics (Courant and Hilbert 1962). This method rests on the fact that the characteristic surface of a set of hyperbolic partial differential equations is determined by the coefficients of the highest derivative terms in the equations. If the normal $n_\mu$ to any characteristic surface happens to be time like, then the wave propagation is acausal. The normals $n_\mu$ are the roots of $D(n)=0$ where $D(n)$ is the characteristic determinant, viz the determinant of the coefficient matrix obtained by replacing $\partial_\mu$ by $n_\mu$ in the highest derivative terms in the true equation of motion.

The equation of motion which we are concerned with is the Kemmer equation (with non-minimal interaction),

$$[\mathrm{i}\beta_4\pi_4+\mathrm{i}\boldsymbol{\beta}\cdot\boldsymbol{\pi}+m+R(x)]\psi=0 \tag{27}$$

where $R(x)$ is given by equation (5). Since $\beta_4$ is singular ($\beta_4^3=\beta_4$) the above equation implies constraints. Projecting to the zero eigenvalue subspace of $\beta_4$ with $P_0=(1-\beta_4^2)$, we get the constraint relation,

$$(1-\beta_4^2)(\mathrm{i}\boldsymbol{\beta}\cdot\boldsymbol{\pi}+m+R)\psi=0$$

which on using $(1-\beta_4^2)\beta_i=\beta_i\beta_4^2$, can be rewritten as

$$[\mathrm{i}\boldsymbol{\beta}\cdot\boldsymbol{\pi}\beta_4^2+(1-\beta_4^2)(m+R)]\psi=0$$

i.e.

$$[\mathrm{i}\boldsymbol{\beta}\cdot\boldsymbol{\pi}\beta_4^2+(1-\beta_4^2)R\beta_4^2]\psi+[m+(1-\beta_4^2)R](1-\beta_4^2)\psi=0. \tag{28}$$

This equation expresses $(1-\beta_4^2)\psi$ in terms of the independent components $\beta_4^2\psi$. Combining this equation with equation (27) we obtain, after straightforward manipulation, the true equation of motion,

$$\{[1+m^{-1}(1-\beta_4^2)R]\pi_4+\beta_4\boldsymbol{\beta}\cdot\boldsymbol{\pi}-\boldsymbol{\beta}\cdot\boldsymbol{\pi}\beta_4-m^{-1}\boldsymbol{\beta}\cdot\boldsymbol{\pi}\beta_4R$$
$$-\mathrm{i}\beta_4(m+R)-m^{-1}(1-\beta_4^2)\partial R/\partial t+m^{-1}\mathrm{i}\boldsymbol{\beta}\cdot\boldsymbol{E}\beta_4^2\}\psi=0. \tag{29}$$

Equation (29) together with the constraint, equation (28), is equivalent to the equation of motion (27).

Replacing $\pi_\mu$ by $n_\mu$ in the first order derivative terms in (29) we obtain the characteristic determinant to be

$$D(n)=\|\{[1+m^{-1}(1-\beta_4^2)R]n_4+\beta_4\boldsymbol{\beta}\cdot\boldsymbol{n}-\boldsymbol{\beta}\cdot\boldsymbol{n}\beta_4-m^{-1}\boldsymbol{\beta}\cdot\boldsymbol{n}\beta_4R\}\|. \tag{30}$$

To evaluate (30), let us specialise to the frame in which $n_\mu=(0,0,0,n_4)$: In this case $D(n)$ simplifies to

$$D(n_4)=n_4^{10}\|1+m^{-1}(1-\beta_4^2)R\|. \tag{31}$$

Substitution of the form of $R$ from (5) and a straightforward evaluation of the determinant leads to

$$D(n_4) = n_4^{10} D_1(n_4) D_2(n_4) \tag{32}$$

with

$$D_1(n_4) = x^3 + \frac{x^2}{m^4}(\alpha_3 \boldsymbol{E}^2 - \rho_3 \boldsymbol{H}^2) - x\left[\frac{\rho_2^2 \boldsymbol{H}^2}{m^4} + \frac{\alpha_3\rho_3}{m^8}\boldsymbol{E}^2\boldsymbol{H}^2 - (\boldsymbol{E}\cdot\boldsymbol{H})^2\right]$$
$$+\frac{\rho_2^2\rho_3}{m^8}(\boldsymbol{H}^2)^2 - \frac{\alpha_3\rho_2^2}{m^8}(\boldsymbol{E}\cdot\boldsymbol{H})^2 \tag{33a}$$

and

$$D_2(n_4) = 1 + \frac{\sigma_1}{m^4}\boldsymbol{H}^2 - \frac{1}{m^4}(\sigma_1 + \sigma_3)\boldsymbol{E}^2. \tag{33b}$$

In the above equation

$$x = 1 + \frac{(\alpha_1 + \alpha_3)}{2m^4} F_{\mu\alpha}F_{\mu\alpha}. \tag{34}$$

The covariant form of the characteristic determinant for the case of arbitrary $n_\mu$ is easily seen to be obtained through the replacements

$$n_4^4 D_1(n_4) \Rightarrow \mathscr{D}_1(n) = (n^2)^2\left[x^3 + \frac{1}{16m^8}\alpha_3(\rho_3 x - \rho_2^2)(\hat{\boldsymbol{F}}\cdot\boldsymbol{F})^2\right]$$
$$+\frac{n^2}{m^4}[(\rho_3 x^2 + \rho_2^2 x)(\hat{\boldsymbol{F}}\cdot\boldsymbol{n})^2 - \alpha_3 x^2(\boldsymbol{F}\cdot\boldsymbol{n})^2] + \frac{\rho_2^2\rho_3}{m^8}(\hat{\boldsymbol{F}}\cdot\boldsymbol{n})^2(\hat{\boldsymbol{F}}\cdot\boldsymbol{n})^2$$
$$-\frac{\alpha_3\rho_3 x}{m^8}(\hat{\boldsymbol{F}}\cdot\boldsymbol{n})^2(\boldsymbol{F}\cdot\boldsymbol{n})^2 \tag{35}$$

and

$$n_4^2 D_2(n_4) \Rightarrow \mathscr{D}_2(n) = n^2 - \frac{\sigma_1}{m^4}(\hat{\boldsymbol{F}}\cdot\boldsymbol{n})^2 + \frac{1}{m^4}(\sigma_1 + \sigma_3)(\boldsymbol{F}\cdot\boldsymbol{n})^2 \tag{36}$$

where we have defined

$$\hat{F}_{\mu\nu} = \tfrac{1}{2}\mathrm{i}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}; \qquad \hat{\boldsymbol{F}}\cdot\boldsymbol{F} = \hat{F}_{\mu\nu}F_{\mu\nu}$$
$$(\hat{\boldsymbol{F}}\cdot\boldsymbol{n})^2 = (\hat{\boldsymbol{F}}\cdot\boldsymbol{n})_\mu(\hat{\boldsymbol{F}}\cdot\boldsymbol{n})_\mu = \hat{F}_{\mu\nu}\hat{F}_{\mu\rho}n_\nu n_\rho \text{ etc.} \tag{37}$$

Thus the form of the characteristic determinant for arbitrary $n_\mu$ is given by

$$D(n) = (n^2)^2\mathscr{D}_1(n)\mathscr{D}_2(n). \tag{38}$$

The normals to the characteristic surfaces (other than the light cone) are solutions of

$$\mathscr{D}_1(n) = 0$$

and

$$\mathscr{D}_2(n) = 0. \tag{39}$$

We shall obtain the solutions of (39) for the following special cases
($a$) $F_{ij} \neq 0, F_{i4} = 0$ (pure magnetic field)
($b$) $F_{i4} \neq 0, F_{ij} = 0$ (pure electric field).

(*a*) *Pure magnetic field*: For simplicity let us take the form of $n_\mu$ to be $(n_1, 0, 0, n_4)$, and take the magnetic field to be along the $z$ direction. It is easily seen that the solutions of (39) are given by

$$-\frac{n_4^2}{n_1^2}=\frac{1+m^{-4}(\sigma_1+\sigma_3)H^2}{1+m^{-4}\sigma_1 H^2}$$

$$-\frac{n_4^2}{n_1^2}=\frac{1+m^{-4}(\rho_1+\rho_3)H^2}{1+m^{-4}\rho_1 H^2}$$

and

$$-\frac{n_4^2}{n_1^2}=\frac{[1+m^{-4}(\rho_1+\rho_3)H^2][1+m^{-4}\alpha_1 H^2]}{[1+m^{-4}(\rho_1+\rho_3)H^2]^2-m^{-4}\rho_2^2H^2}. \tag{40}$$

From the above equations it may be easily verified that the waves propagate causally, provided

$$\alpha_1, \sigma_1, \rho_1, (\sigma_1+\sigma_3), (\rho_1+\rho_3)\geqslant 0$$
$$\rho_3, \sigma_3 \leqslant 0 \quad \text{and} \quad (\alpha_1+\alpha_3)\geqslant \tfrac{1}{4}\alpha_2^2. \tag{41}$$

One can check that there exist overlapping regions between the conditions (40) for causality and the conditions (26) for real eigenvalues in a HMF.

(*b*) *Pure electric field*: Let us choose the electric field to be along the $z$ direction and as before consider the special frame in which $n_\mu=(n_1, 0, 0, n_4)$. For this case the solutions for $n_4^2$ are again easily calculated.

$$-\frac{n_4^2}{n_1^2}=\frac{1-m^{-4}\sigma_1 E^2}{1-m^{-4}(\sigma_1+\sigma_3)E^2}$$

$$-\frac{n_4^2}{n_1^2}=\frac{1-m^{-4}\rho_1 E^2}{1-m^{-4}(\rho_1+\rho_3)E^2} \tag{42}$$

and

$$-\frac{n_4^2}{n_1^2}=\frac{[1-m^{-4}(\alpha_1+\alpha_3)E^2]^2+m^{-4}\alpha_2^2E^2}{[1-m^{-4}(\alpha_1+\alpha_3)E^2][1-m^{-4}\alpha_1 E^2]}.$$

One can verify that causal propagation in this case demands that

$$\alpha_1, \sigma_1, \rho_1, (\sigma_1+\sigma_3), (\rho_1+\rho_3)\leqslant 0$$
$$\rho_3, \sigma_3 \geqslant 0 \tag{43}$$

and

$$(\alpha_1+\alpha_3)\leqslant \tfrac{1}{4}\alpha_2^2.$$

The conditions (41) and (43) for causal propagation in a pure magnetic and a pure electric field respectively are mutually compatible only if the equality sign is taken in all cases. Thus, propagation in both pure magnetic and pure electric fields is caused only if

$$\alpha_1=\rho_1=\rho_3=\sigma_1=\sigma_3=0 \tag{44a}$$

and

$$\alpha_2 \equiv \rho_2 = 0. \tag{44b}$$

Equations (44$a$), translated to the original coupling constants, immediately show that

$$g_3 = g_4 = g_5 = g_6 = 0. \tag{45}$$

That is, causality demands that *all* the quadratic terms in the anomalous coupling term should go. From (44$b$), we find that $g_1 + 4g_2 = 0$: i.e.,

$$g_1 = -4g_2.$$

One can verify from (35) and (37), for completeness, that with the conditions (44) and (46), causality of propagation is retained for *arbitrary* electromagnetic fields $F_{\mu\nu}$.

## 6. The Proca and Shay–Good forms

For the above choice of the anomalous coupling (equations (45) and (46)) the equations of motion for $\psi_i$ take the form ($\kappa = 2g_2$):

$$\begin{aligned}
&-m\psi_1 = \pi_4\psi_3 - \mathrm{i}\boldsymbol{e} \cdot \boldsymbol{\pi}\psi_4 \\
&-m\psi_2 = \boldsymbol{\sigma} \cdot \boldsymbol{\pi}\psi_3 \\
&\pi_4\psi_1 + \boldsymbol{\sigma} \cdot \boldsymbol{\pi}\psi_2 - m\psi_3 + \frac{e\kappa}{m}\boldsymbol{\sigma} \cdot \boldsymbol{H}\psi_3 + \frac{\mathrm{i}e\kappa}{m}\boldsymbol{e} \cdot \boldsymbol{E}\psi_4 = 0 \\
&\mathrm{i}\boldsymbol{e}^\dagger \cdot \boldsymbol{\pi}\psi_1 - m\psi_4 + \frac{\mathrm{i}e\kappa}{m}\boldsymbol{e}^\dagger \cdot \boldsymbol{E}\psi_3 = 0.
\end{aligned} \tag{47}$$

The first two equations can be taken to define $\psi_1$ and $\psi_2$ in terms of $\psi_3$ and $\psi_4$. On eliminating these from the last two equations, we get

$$\begin{aligned}
&[\pi_4^2 + m^2 + (\boldsymbol{\sigma} \cdot \boldsymbol{\pi})^2 - e\kappa\boldsymbol{\sigma} \cdot \boldsymbol{H}]\psi_3 - \mathrm{i}[\pi_4\boldsymbol{e} \cdot \boldsymbol{\pi} + e\kappa\boldsymbol{e} \cdot \boldsymbol{E}]\psi_4 = 0 \\
&\qquad \mathrm{i}[\boldsymbol{e}^\dagger \cdot \boldsymbol{\pi}\pi_4 - e\kappa\boldsymbol{e}^\dagger \cdot \boldsymbol{E}]\psi_3 + [\boldsymbol{e}^\dagger \cdot \boldsymbol{\pi}\boldsymbol{e} \cdot \boldsymbol{\pi} + m^2]\psi_4 = 0.
\end{aligned} \tag{48}$$

With the identification of the 3-components of $\psi_3$ with a vector $\boldsymbol{V}$ and $\psi_4$ with $-\mathrm{i}V_4$, we can rewrite the above equations in the vector form

$$\pi_\mu(\pi_\mu V_\nu - \pi_\nu V_\mu) + m^2 V_\nu + \mathrm{i}e\kappa F_{\nu\mu} V_\mu = 0 \tag{49}$$

which is just the Proca equation with an anomalous dipole term. Our results verify that the Proca equation with this coupling is indeed causal irrespective of the strength.

More interesting is the generalised Shay–Good form obtained by eliminating $\psi_3$ and $\psi_4$ from the equations of motion (47) and writing them solely in terms of $\psi_1$ and $\psi_2$, which together constitute an antisymmetric tensor. One has

$$-m^2\begin{bmatrix}\psi_1 \\ \psi_2\end{bmatrix} = (\pi_4 - \mathrm{i}\boldsymbol{e} \cdot \boldsymbol{\pi})\begin{bmatrix} 1 - \frac{e\kappa}{m^2}\boldsymbol{\sigma} \cdot \boldsymbol{H} & -\mathrm{i}\frac{e\kappa}{m^2}\boldsymbol{e} \cdot \boldsymbol{E} \\ -\frac{\mathrm{i}e\kappa}{m^2}\boldsymbol{e}^\dagger \cdot \boldsymbol{E} & 1 \end{bmatrix}^{-1}\begin{bmatrix} \pi_4 & \boldsymbol{\sigma} \cdot \boldsymbol{\pi} \\ \mathrm{i}\boldsymbol{e}^\dagger \cdot \boldsymbol{\pi} & 0 \end{bmatrix}\begin{bmatrix}\psi_1 \\ \psi_2\end{bmatrix} \tag{50}$$

or, in tensor language, ($\psi_1 = -\mathrm{i}G_{j4}$, $\psi_2 = \frac{1}{2}\epsilon_{ijk}G_{jk}$)

$$m^2 G_{\mu\nu} = \pi_\mu\left(1 + \mathrm{i}\frac{e\kappa}{m^2}F\right)^{-1}_{\nu\alpha}\pi_\rho G_{\alpha\rho} - (\mu \leftrightarrow \nu). \tag{51}$$

(Here $F$ is a $4\times 4$ antisymmetric matrix with elements $F_{\mu\nu}$). It is not difficult to evaluate the inverse. We find

$$\left(1+\frac{ie\kappa}{m^2}F\right)^{-1}_{\mu\nu}$$

$$=\frac{(1+\frac{1}{2}(e\kappa/m^2)^2F^2)\delta_{\mu\nu}-(ie\kappa/m^2)F_{\mu\nu}-\frac{i}{4}(e\kappa/m^2)^3(\hat{\boldsymbol{F}}\,.\,\boldsymbol{F})\hat{F}_{\mu\nu}-(e\kappa/m^2)^2F_{\mu\nu}F_{\mu\nu}}{1+\frac{1}{2}(e\kappa/m^2)^2F^2-\frac{1}{16}(e\kappa/m^2)^4(\hat{\boldsymbol{F}}\,.\,\boldsymbol{F})^2}$$

where $\hat{F}_{\mu\nu}$ is the tensor dual to $F_{\mu\nu}$ (equation 37). It is clear that the inverse matrix brings in terms which are explicitly non-linear in the $F_{\mu\nu}$ (with coefficients which themselves depend in non-polynomial fashion on invariants constructed from the $F_{\mu\nu}$). Further, the action of $\pi_\mu$ on this matrix introduces a dependence of the interaction on derivatives of the $F_{\mu\nu}$, i.e. quadrupole interaction.

This observation enables us to explain a result which had remained puzzling, namely that in the Shay–Good theory with non-minimal electromagnetic interaction introduced through a Pauli type term, causality is possible only if the strength of the non-minimal term is so chosen as to make the $g$-factor equal to unity, while in the Proca theory with similar non-minimal interaction, causality persists whatever be the anomalous magnetic moment. We see now that the latter, translated into the Shay–Good language has interaction terms which are explicitly non-linear in the $F_{\mu\nu}$ and have strengths which are themselves field dependent. What is more pertinent, the structure of the non-minimal interaction term in (51), namely

$$\pi_\mu\left(\left[1+\frac{ie\kappa}{m^2}F\right]^{-1}-1\right)\pi_\rho G_{\alpha\rho}$$

is quite different from the form $(ie/2)(1+\kappa)(F_{\lambda\mu}G_{\nu\lambda}-F_{\nu\lambda}G_{\lambda\mu})$ assumed by Shay and Good: it is not the $G_{\mu\nu}$ which occur in the former, but their second derivatives $\pi_\mu\pi_\rho G_{\alpha\rho}$. What we learn is that to preserve causality for arbitrary values of the magnetic moment, the non-minimal interaction has to be introduced into the Shay–Good equation through the $\pi_\mu\pi_\rho G_{\alpha\rho}$ term in the manner of equation (51), and not through the 'natural' looking term actually used by Shay and Good.

## 7. Results and discussion

We see from equations (45) and (46) that only terms linear in $F_{\mu\nu}$ are permissible in the DKP equation if causality of propagation is to be ensured. It is interesting to note that the 'Pauli term' $\sigma_{\mu\nu}F_{\mu\nu}$ taken by itself leads to violation of causality; an appropriate admixture of this term with $\{B,\sigma_{\mu\nu}\}F_{\mu\nu}$ has to be taken to eliminate this trouble. This result brings out clearly the fact that the 'natural' appearance of some interaction terms (such as the Pauli term) is by no means sufficient to ensure satisfactory results, and emphasises the need for the investigation of interactions in all generality. Furthermore, the degree of generality needed may be well beyond what one might ordinarily think of, as is strikingly illustrated by the example of the equation in Shay–Good form.

As regards the reality of the energy spectrum in the presence of a homogeneous magnetic field, we have found in § 4 that if non-minimal interaction terms linear in the $F_{\mu\nu}$ are present, then the spectrum can be wholly real only if other terms quadratic in the $F_{\mu\nu}$ are also present with sufficient strength. Combining this with the condition for causality, noted above, we conclude that causality and the reality of eigenvalues can both be obtained only if the coupling is restricted to be minimal. (This corresponds to the $g$-factor being unity). However, it has long been known that the minimal coupling has its own pathology: Corben and Schwinger (1940) showed that with minimal coupling a complete set of wavefunctions having acceptable behaviour does not exist for a spin-one particle in a Coulomb field.

The present investigation is the first, as far as the authors are aware, to consider the direct interactions of the DKP particle and to study the problem of the energy spectrum, besides the causality problem, in this general context. It encompasses all earlier studies with specific anomalous terms. It is disconcerting to find that despite the generality a completely consistent possibility has not emerged.

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